A Comparison of a Relaxation Solver and a Preconditioned Conjugate Gradient Solver in Parallel Finite Element Groundwater Computations

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Governing Equations

$$\nabla \cdot \left(\mathbf{k} \cdot \nabla \phi \right) = 0$$

$\mathbf{k}$ = hydraulic conductivity tensor

ϕ = potential or total head

Linear System

$$[K]\{\phi\} = \{Q\}$$

[K] = stiffness matrix

{ϕ} = vector of total head (L) values at the nodes

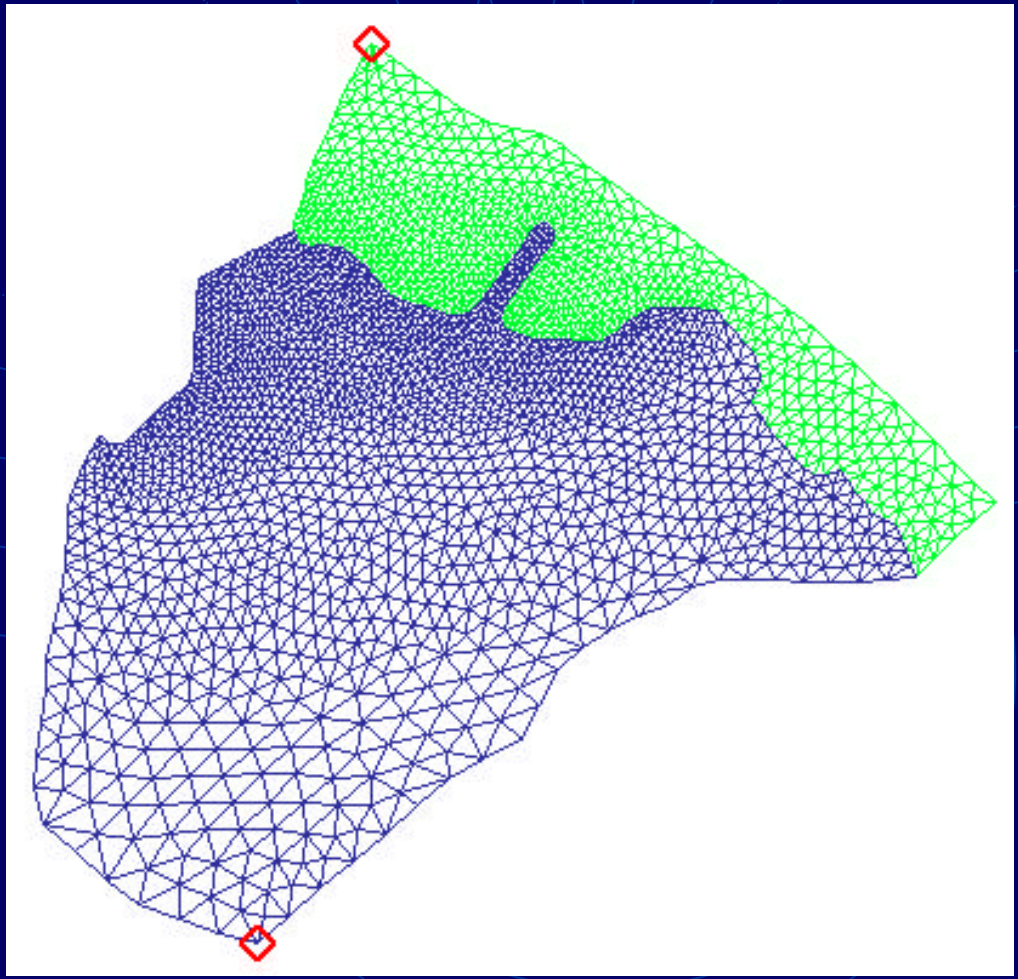
{Q} = vector of external flow (L^3/T) values at the nodes

Relaxation
Forward and Backward Passes

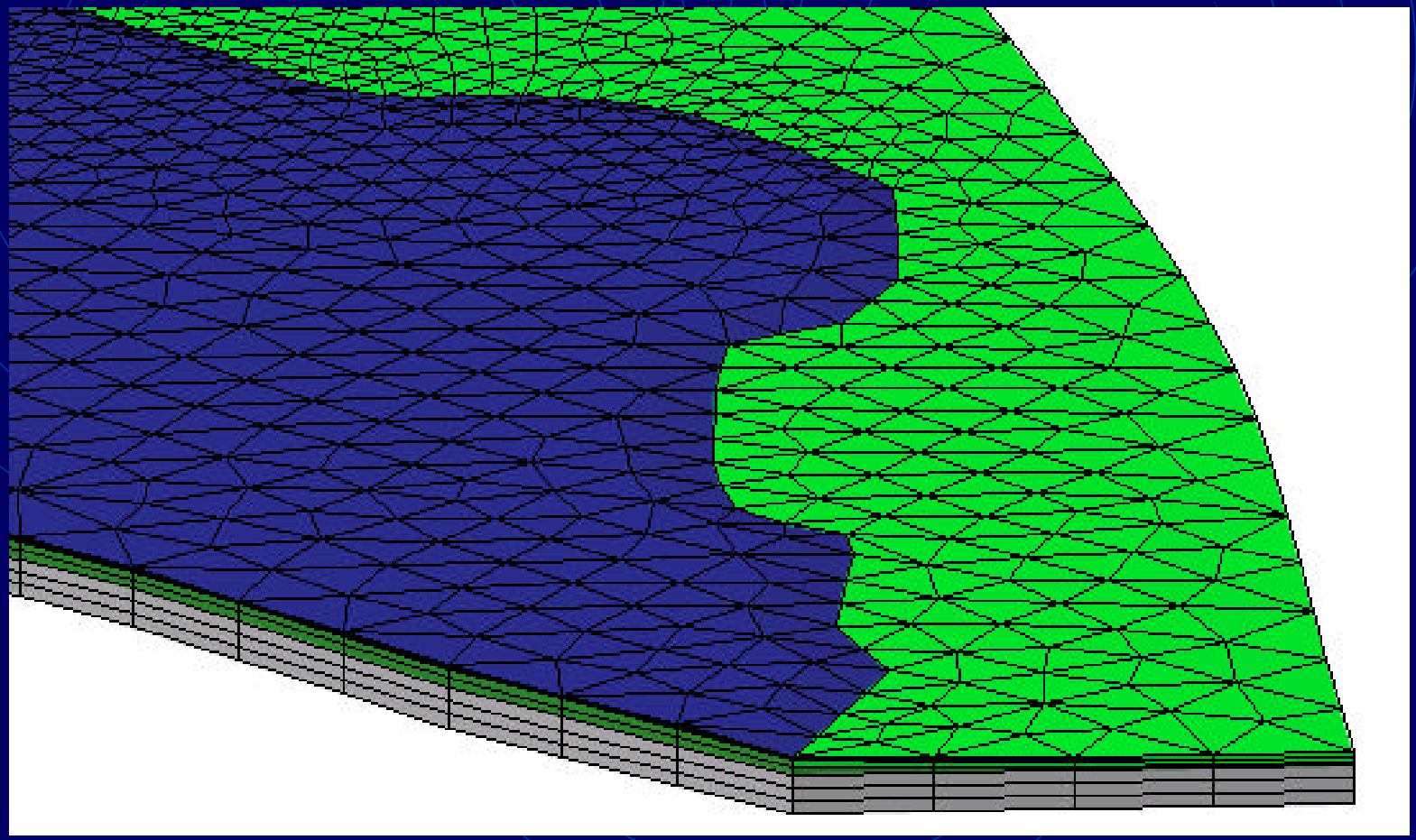
$$\delta\phi_i^{new} = \frac{\left(Q_i - \sum_j K_{ij}\phi_j\right)}{K_{ii}}$$

$$\phi_i^{new} = \phi_i^{old} + \varpi\delta\phi_i^{new}$$

Plan View
18,752 Nodes – 31,787 Elements

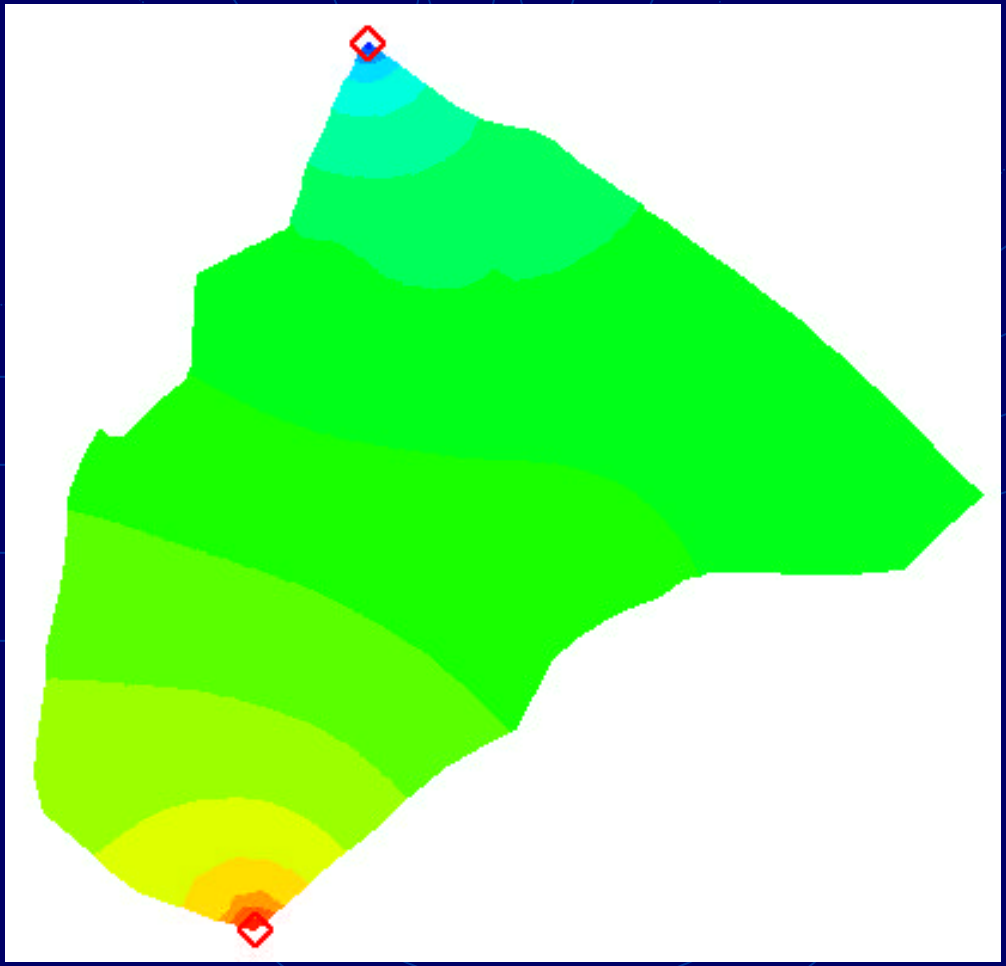


Isometric View

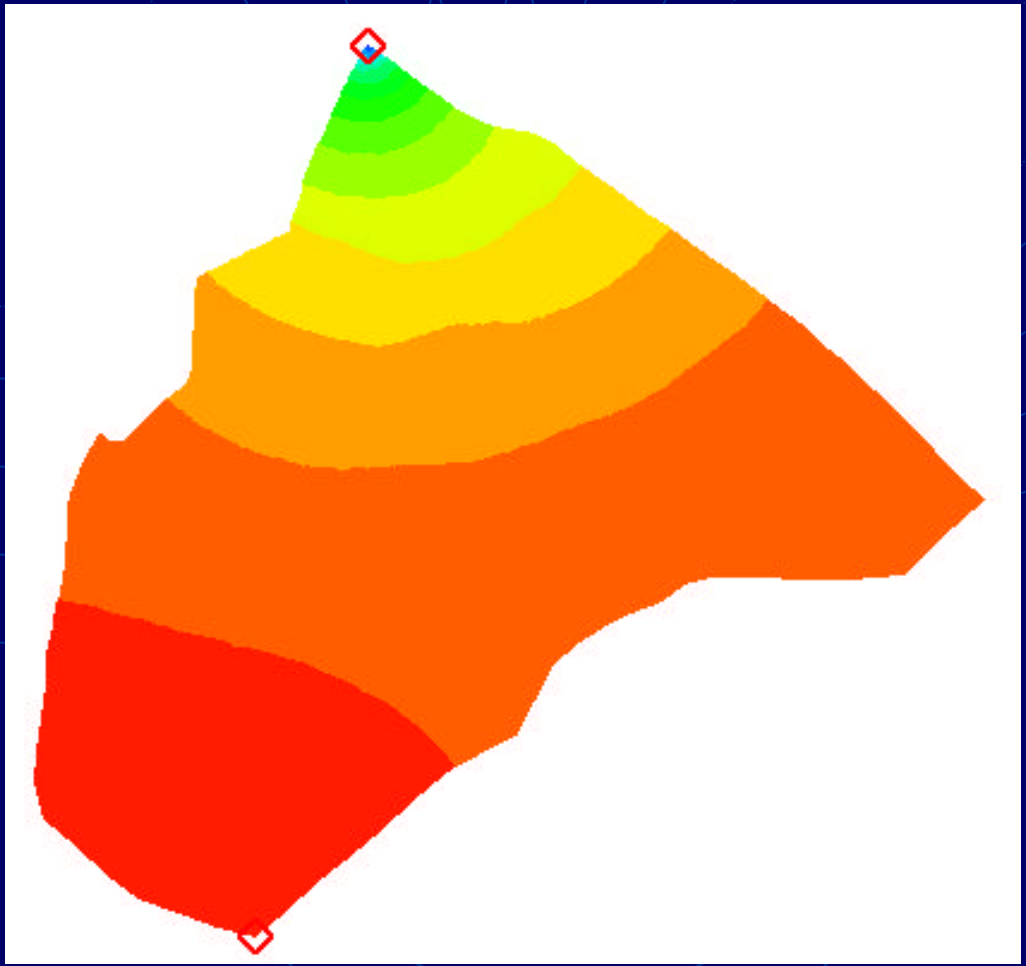


Isolevels – ILU CG

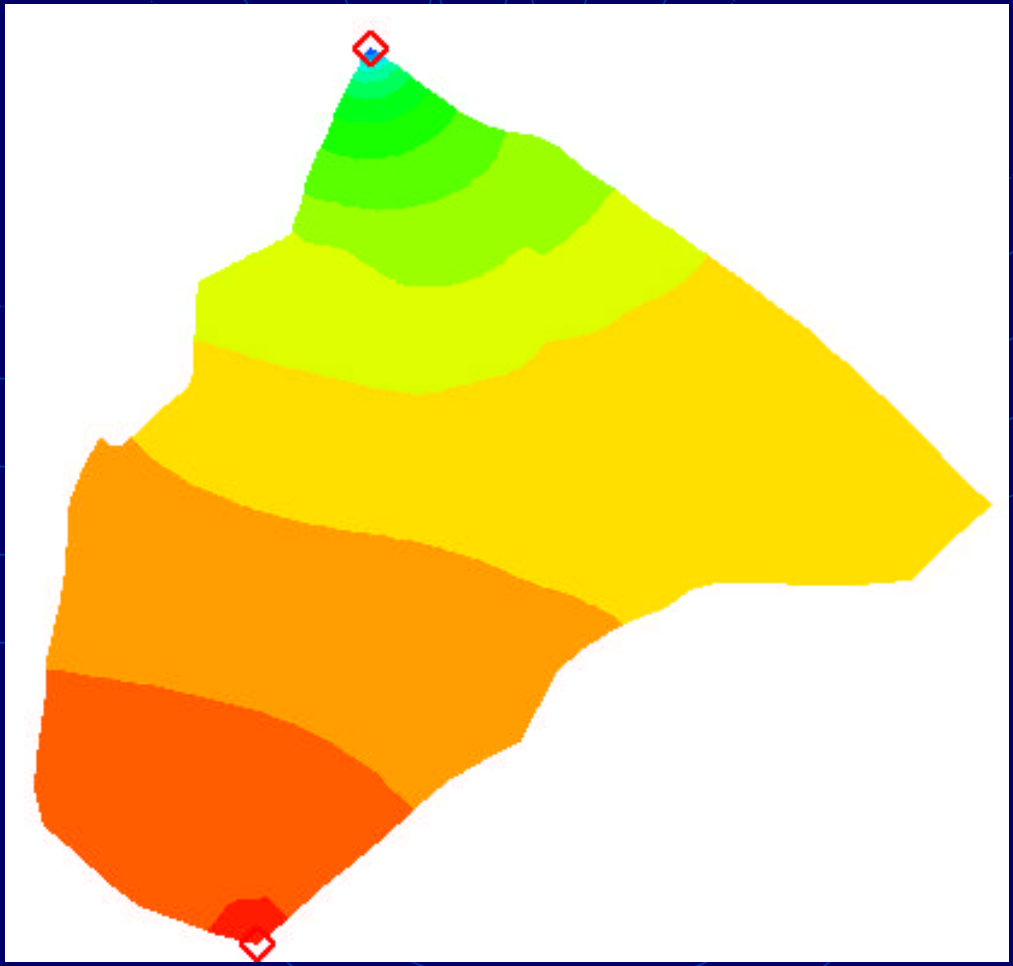
ILU CG – 341 Iterations



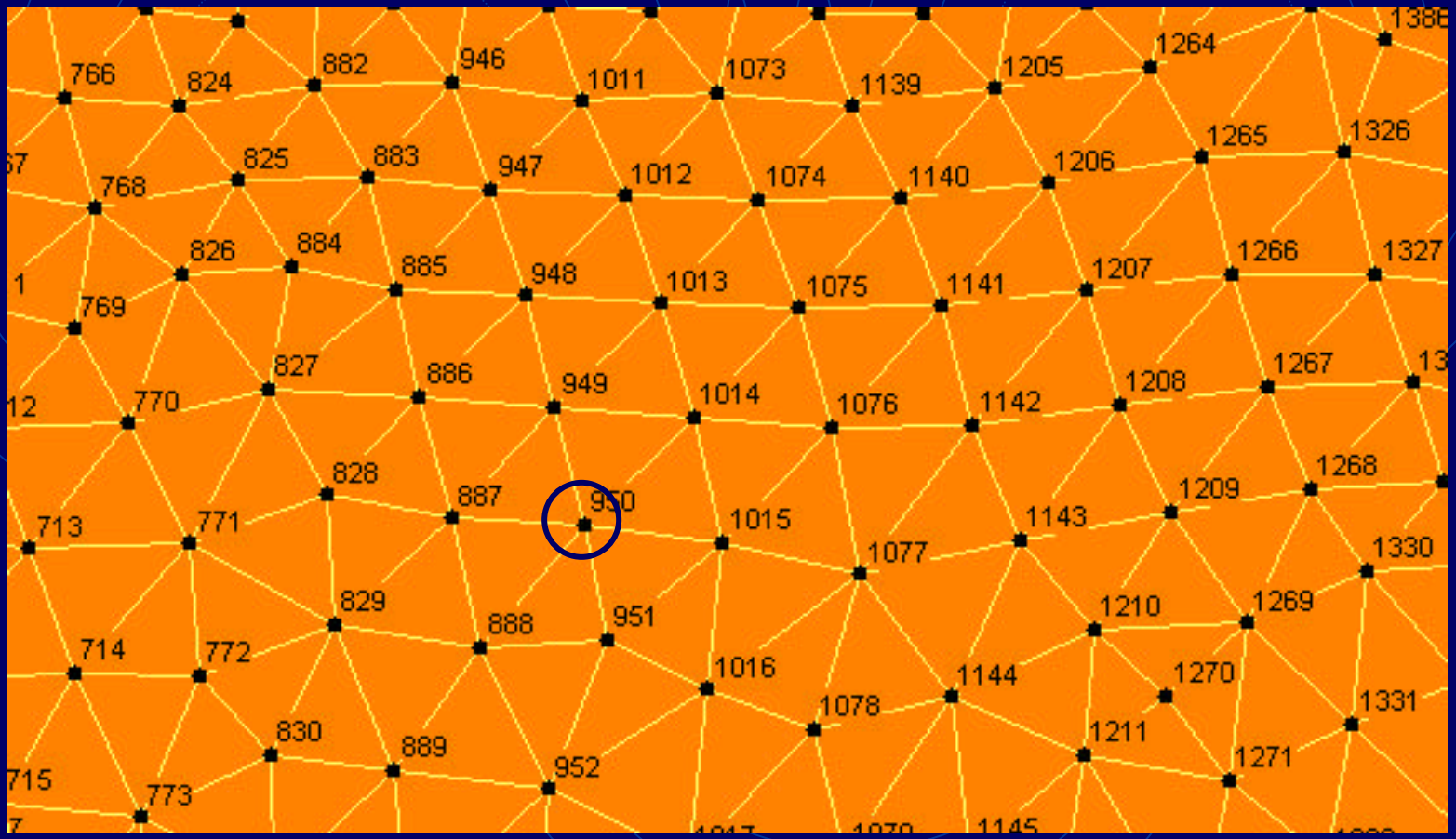
Relaxation – 5,000 Iterations



Relaxation – 10,000 Iterations



Node 950


766
824
882
946
1011
1073
1139
1205
1264
1386
768
825
883
947
1012
1074
1140
1206
1265
1326
769
826
884
885
948
1013
1075
1141
1207
1266
1327
770
827
886
949
1014
1076
1142
1208
1267
713
771
828
887
950
1015
1077
1143
1209
1268
1330
714
772
829
888
951
1016
1078
1144
1210
1269
1270
1331
715
773
830
889
952
1211
1271

Relaxation Results
Specified Head

Iterations	Max. Abs. Error	Pressure Head CG = 144.4	% Diff.
10,000	0.001818	173.1	18.1
20,000	0.000960	161.3	11.1
30,000	0.000590	154.4	6.7
40,000	0.000347	150.0	3.8
50,000	0.000205	147.9	2.4

Relaxation Results
Specified Flow

Iterations	Max. Abs. Error	Pressure Head CG = 144.4	% Diff.
10,000	0.000836	184.7	17.2
20,000	0.000629	178.0	13.5
30,000	0.000482	172.8	10.5
40,000	0.000369	168.8	8.2
50,000	0.000290	165.8	6.4

ILU CG Preconditioner

$$\left[\tilde{K}\right] = \left[\varpi L + D\right]\left[D\right]^{-1}\left[D + \varpi U\right]$$

[L] = lower part of [K]

[D] = diagonal part of [K]

[U] = upper part of [K]

ϖ = relaxation type factor

Two Systems of Equations

$$\left[\tilde{K}\right]\{z\} = \{r\} = \{Q\} - [K]\{\phi\}$$

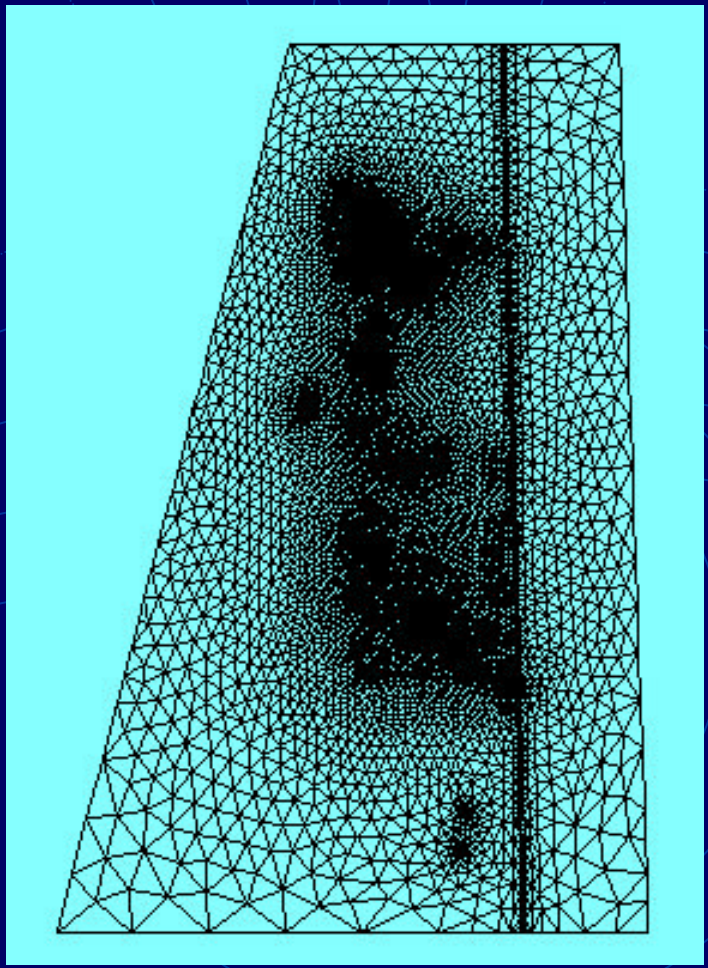
$$[D + \varpi U]\{y\} = \{r\}$$

$$[\varpi L + D]\{z\} = [D]\{y\}$$

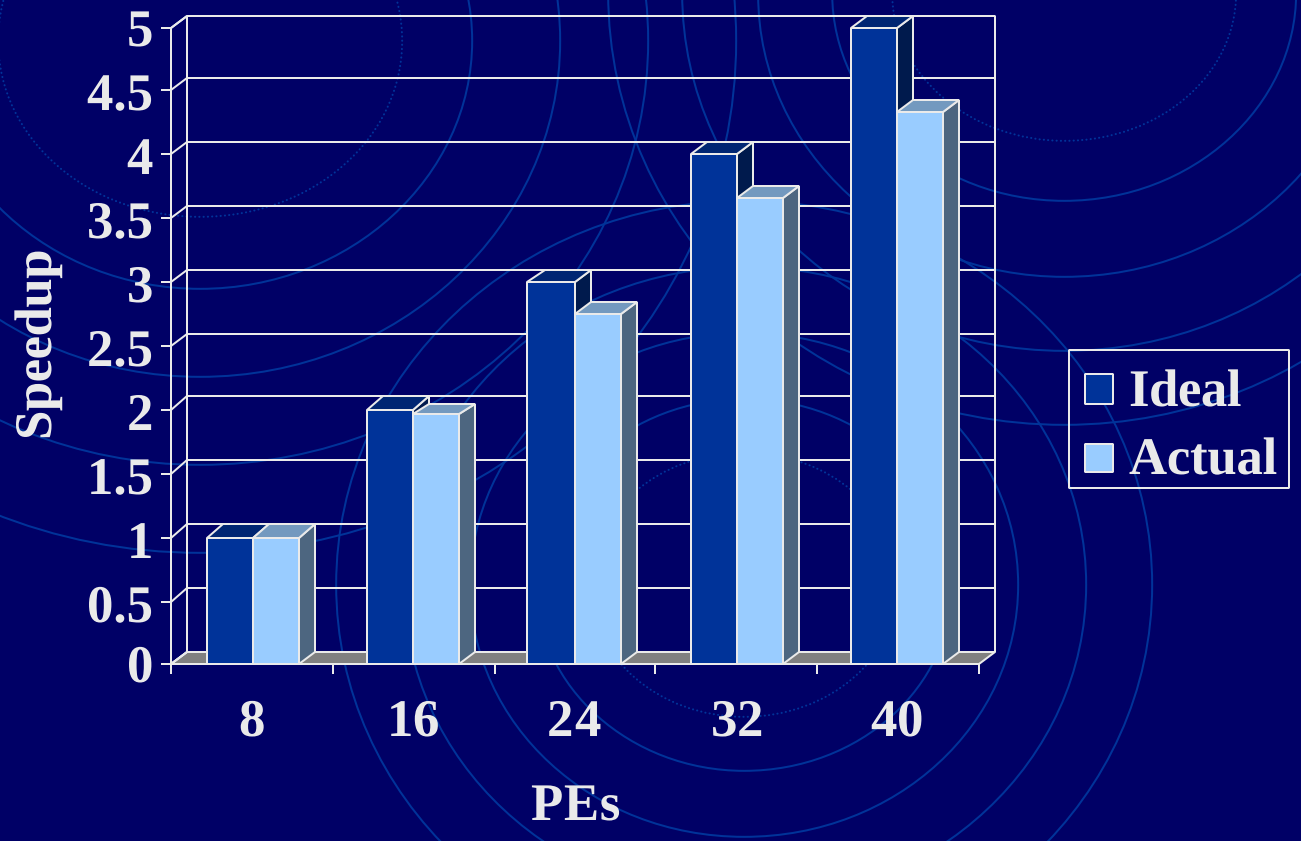
Parallel Implementation

$$
\begin{array}{l}
\text{PE 0} \\ \\ \\ \\ \\ \\
\text{PE 1} \\ \\ \\ \\
\end{array}
\left[\begin{array}{ccccc|cccc}
* & * & 0 & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & 0 & 0 & 0 \\
0 & * & * & 0 & * & * & 0 & 0 & 0 \\
* & * & 0 & * & * & 0 & * & * & 0 \\
* & * & * & * & * & * & * & * & * \\
\hline
0 & * & * & 0 & * & * & 0 & * & * \\
0 & 0 & 0 & * & * & 0 & * & * & 0 \\
0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & 0 & * & *
\end{array}\right]
\{y\} = \{r\}
$$

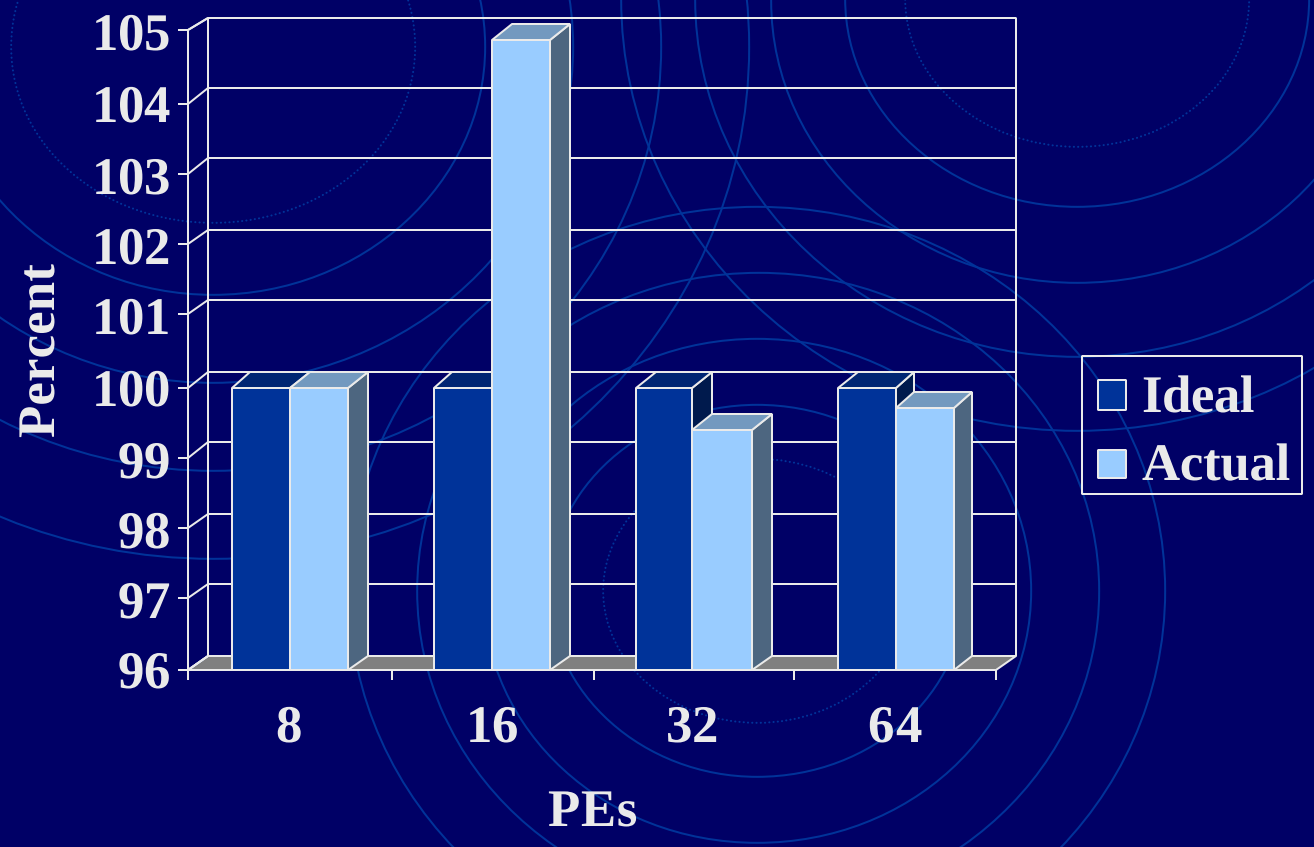
102,996 Nodes
187,902 Elements



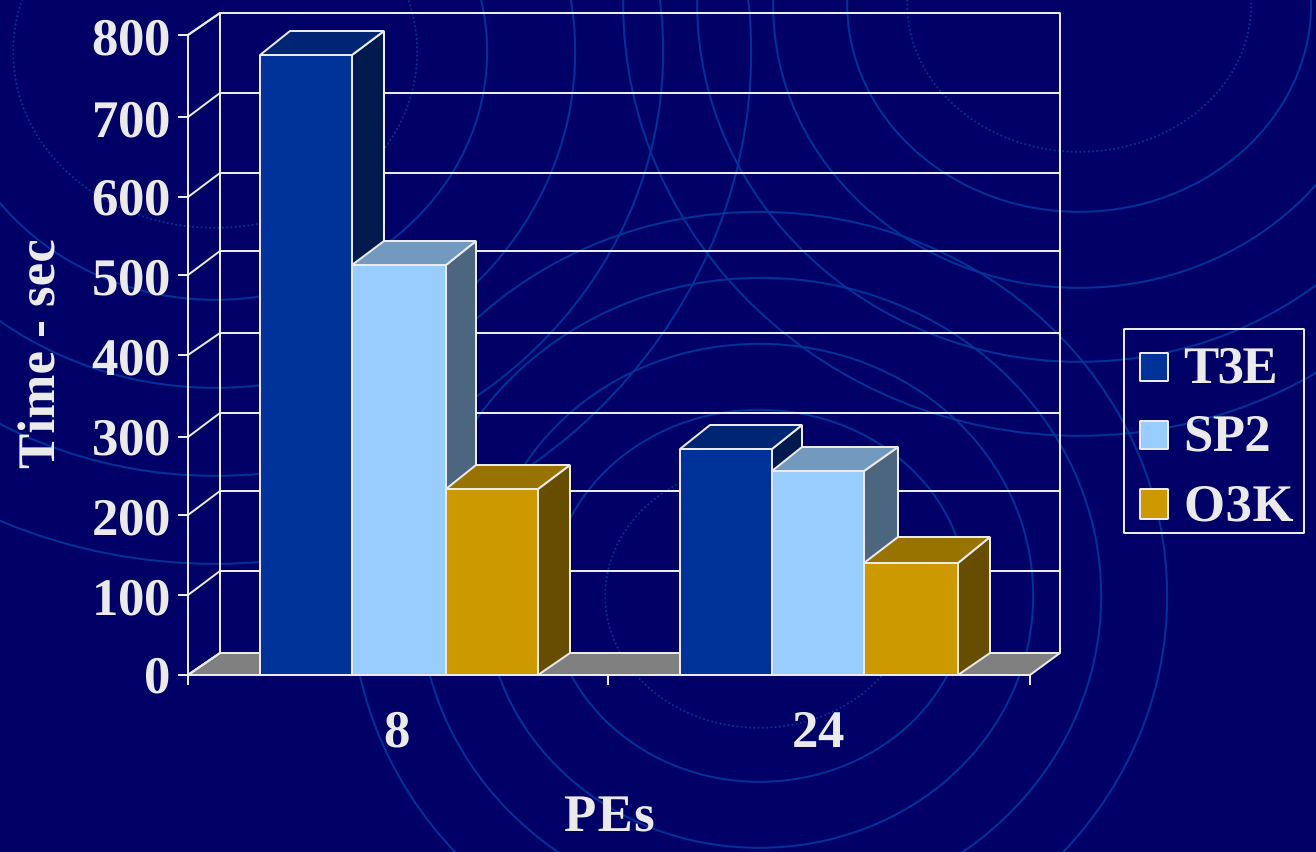
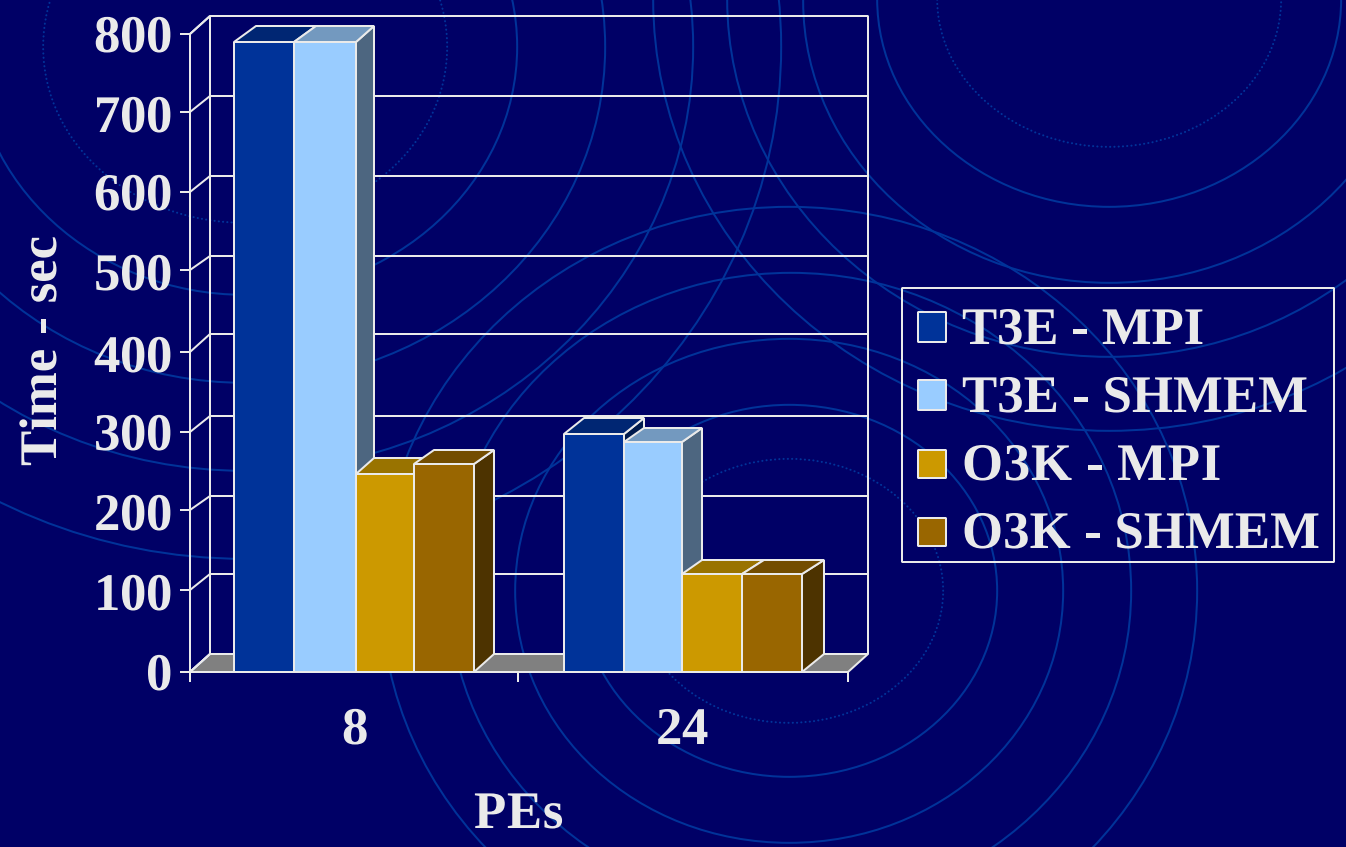
Parallel Speedup – T3E
8 PE Run – 777.3 sec



Scaled Speedup – T3E
8 PE Run – 777.3 sec



Comparison of Systems



MPI Versus SHMEM